

LINEAR HOLONOMY OF MARGULIS SPACE-TIMES

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To Aimee, with love

1. Introduction: Free discrete groups

If $\Gamma \subset \text{Aff}(\mathfrak{R}^3)$ acts properly discontinuously on \mathfrak{R}^3 , then Γ is either solvable or free up to finite index [3], [6]. If Γ is free and acts properly discontinuously on \mathfrak{R}^3 , then Γ is conjugate to a subgroup of $\mathbf{H} = \mathbf{O}(2, 1) \ltimes \mathbf{V}$, where \mathbf{V} is the group of parallel translations in $\mathbf{E} = \mathfrak{R}^{2,1}$ [3]. Let $\mathbf{G} = \text{SO}(2, 1)$ and let \mathbf{G}^o denote its identity component.

Complete affinely flat manifolds correspond to $\Gamma \subset \text{Aff}(\mathfrak{R}^3)$ which act properly discontinuously and freely on \mathbf{E} . Define *Margulis space-times* as complete affinely flat 3-dimensional manifolds with free fundamental group; their existence was demonstrated by Margulis [4], [5].

Let $L: \text{Aff}(\mathfrak{R}^3) \rightarrow \text{GL}(n, \mathfrak{R})$ be the usual projection. If Γ acts properly discontinuously on \mathbf{E} , then $L(\Gamma)$ is conjugate to a free discrete group of \mathbf{G} ; it was shown in [2].

Theorem 1. *For every Schottky group $G \subset \mathbf{G}^o$ there exists a free $\Gamma \subset \mathbf{H}$ which acts properly discontinuously on \mathbf{E} and $L(\Gamma) = G$.*

$G \subset \mathbf{G}^o$ is a Schottky group if and only if all nonidentity elements are hyperbolic. The set of all Schottky groups in \mathbf{G}^o is a proper subset of the set of all free discrete subgroups of \mathbf{G}^o . In particular, there are free discrete subgroups of \mathbf{G}^o , which contain parabolic elements.

We shall prove

Theorem 2. *$G = L(\Gamma)$ for some free finitely generated $\Gamma \subset \text{Aff}(\mathfrak{R}^3)$ which acts properly discontinuously on \mathbf{E} if and only if G is conjugate to a free finitely generated discrete subgroups of \mathbf{G} .*

For the affine manifold \mathbf{M} , the group of deck transformations Π acts on the universal cover $\tilde{\mathbf{M}}$ by affine automorphisms. The developing map $D: \tilde{\mathbf{M}} \rightarrow \mathbf{E}$ is a homeomorphism for complete \mathbf{M} . For every $\tau \in \Pi$ there

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is a unique affine automorphism $\phi(\tau)$ such that $D \circ \tau = \phi(\tau) \circ D$, and $\phi: \Pi \rightarrow \text{Aff}(\mathfrak{R}^3)$ is called the *affine holonomy representation*.

$L \circ \phi: \Pi \rightarrow \text{GL}(3, \mathfrak{R})$ is called the *linear holonomy representation*. Margulis conjectured that $L \circ \phi(\Pi)$ for any complete affine flat manifold with free fundamental group contains no parabolic elements. Theorem 2 shows that this conjecture is false. More generally, Theorem 2 is a classification of linear holonomy representations of complete affinely flat manifolds with free fundamental group.

2. Generalized Schottky groups

Classically, Schottky groups lie in $\text{PSL}_2(\mathbb{C})$. We can consider their restriction to $\text{PSL}_2(\mathfrak{R})$. We also allow for orientation reversing matrices and define a *Schottky group* $\mathbf{H} \subset \text{PGL}_2(\mathfrak{R})$ of n generators as a free group which acts properly discontinuously on the hyperbolic plane \mathbf{H}^2 such that there exists a fundamental domain for its action which is bounded by $2n$ complete ultraparallel geodesics.

We define a *generalized Schottky group* of n generators in $\text{PGL}_2(\mathfrak{R})$ to be a free group which acts properly discontinuously on \mathbf{H}^2 such that there exists a fundamental domain for this action, which is bounded by $2n$ complete nonintersecting geodesics.

The following theorem is well known.

Theorem 3. $\Lambda \subset \text{PGL}_2(\mathfrak{R})$ is a free finitely generated discrete subgroup if and only if it is a generalized Schottky group.

Proof. By definition, a generalized Schottky group is a free finitely generated discrete group of $\text{PGL}_2(\mathfrak{R})$.

Conversely, suppose that Λ acts properly discontinuously on \mathbf{H}^2 . Topologically, \mathbf{H}^2/Λ is an n -punctured surface of genus g , where $k = 2g - n - 1$, and k is the rank of Λ . \mathbf{H}^2/Λ may be an unoriented surface.

We perform surgery on this surface along k nonintersecting puncture-to-puncture curves so that the resulting surface is simply connected. $n - 1$ of these surgeries will be from one puncture to another. There will be $2g$ surgeries from a puncture to itself along curves which are not null homotopic.

We may now construct a fundamental domain \mathbf{X} for the action of Λ on \mathbf{H}^2 . \mathbf{X} is bounded by the $2k$ nonintersecting curves corresponding to the k surgeries.

These curves may be straightened to complete nonintersecting geodesics. The region $\tilde{\mathbf{X}}$ in \mathbf{H}^2 bounded by these $2k$ intersecting geodesics is also a

fundamental domain for the action of Λ on \mathbf{H}^2 . Thus, Λ is a generalized Schottky group. q.e.d.

Let $\pi: \mathbf{G} \rightarrow \text{PGL}_2(\mathfrak{R})$ be the usual isomorphism. Generalized Schottky groups and Schottky groups in \mathbf{G} are defined to be the preimages of generalized Schottky groups and Schottky groups in $\text{PGL}_2(\mathfrak{R})$.

Elements of \mathbf{G} act on \mathbf{E} leaving the inner product

$$B(x, y) = x_1y_1 + x_2y_2 - x_3y_3$$

invariant. The associated cross product is

$$x \boxtimes y = \begin{bmatrix} x_2y_3 - x_3 - y_2 \\ x_3y_1 - x_1y_3 \\ x_2y_1 - x_1y_1 \end{bmatrix}.$$

Let $\mathbf{C} = \{x \in \mathbf{E} | B(x, x) = 0\}$ be the cone invariant under the action of all $g \in \mathbf{G}$, and let

$$\mathbf{N} = \{x \in \mathbf{C} | x_3 > 0\}$$

be its upper nappe which is invariant under \mathbf{G}^o . A *conical neighborhood* \mathbf{A} of \mathbf{C} is a subset of \mathbf{C} such that $\mathbf{A} \cap \mathbf{N}$ is c connected, and if $x \in \mathbf{A}$ then $kx \in \mathbf{A}$ for all $k \in \mathfrak{R}$. For $u, v \in \mathbf{A} \cap \mathbf{N} \cap \mathbf{S}^2$, the ordered pair (u, v) is the *bounding vector pair* for \mathbf{A} if $B(u \boxtimes v, x) \geq 0$ for all $x \in \mathbf{A} \cap \mathbf{N}$.

Note that if (u, v) is the bounding vector pair for conical neighborhood \mathbf{A} , then (v, u) is the bounding vector pair for the conical neighborhood $\mathbf{C} - \mathbf{A}$.

For every generalized Schottky group in \mathbf{G}^o there exist a set of independent generators $\{g_1, g_2, \dots, g_n\}$ and corresponding $2n$ conical neighborhoods $j \{A_1^\pm, A_2^\pm, \dots, A_n^\pm\}$ whose interiors are disjoint and

$$g_i(A_i^-) = \text{cl}(\mathbf{C} - A_i^+).$$

Here cl denotes closure in the usual topology.

Let $\rho(x, y)$ denote the Euclidean distance between $x, y \in \mathbf{E}$. \mathbf{S}^2 is the Euclidean unit sphere centered at the origin, and $\mathbf{B}(x, \delta)$ is the Euclidean δ -ball centered at x .

Generalized Schottky groups in \mathbf{G} can contain *parabolic elements*, i.e., unipotent elements, and do contain *hyperbolic elements*. Hyperbolic elements in \mathbf{G} are defined to have 3 distinct real eigenvalues $|\lambda_g| < 1 < |\lambda_g^{-1}|$, whose corresponding eigenvectors are x_g^-, x_g^o , and x_g^+ . We choose the expanding eigenvector x_g^+ and the contracting eigenvector x_g^- so that both are in $\mathbf{N} \cap \mathbf{S}^2$. The invariant vector x_g^o is chosen so that $B(x_g^o, x_g^o) = 1$ and $\{sx_g^o, x_g^-, x_g^+\}$ is a right-handed basis for \mathbf{E} .

For any $y \in E$ such that $B(y, y) > 0$, let x_y^- and x_y^+ be the elements of $N \cap S^2$ such that $B(y, x_y^\pm) = 0$ and $\{y, x_y^-, x_y^+\}$ is a right-handed basis of E .

g is ε -hyperbolic if it is hyperbolic and $\rho(x_g^-, x_g^+) > \varepsilon$.

Lemma 1. *If $G \subset \mathbf{G}$ is a generalized Schottky group with generators g_1, g_2, \dots, g_n , then for $g \in \Gamma$*

$$x_g^+ \in A_{i_1}^{\text{sign}(j_1)} \quad \text{and} \quad x_g^- \in A_{i_m}^{-\text{sign}(j_m)},$$

where $g = \prod_{k=1}^n (g_{i_k}^{j_k})$ such that $i_k \in \{1, 2, \dots, n\}$, $j_k \in \mathbf{Z} - \{0\}$, and $i_k \neq i_{k+1}$.

Proof. Consider the action of $g' = \pi(g)$ on the boundary of the hyperbolic plane ∂H^2 . Write the projectivization of all $x \in \mathbf{C}$ and $A \subset \mathbf{C}$ as x' and A' , respectively. Note that if $(g^{-1})' = f'$ then $(x_g^-)' = (x_f^+)'$, and if $d' = (g_i^n)'(g)'(g_i^{-n})'$ then $(x_d^+) = (g_i^n)'(x_g^+)'$. Thus, we only need to consider x_g^+ where g is such that $i_1 \neq i_n$. By an appropriate change of generators we have $j_1 > 0$ and $i_1 = 1$.

We can show by induction that $g'((A_1^+)') \subset (A_1^+)'$. Hence Brouwer's fixed point theorem shows that g' has a fixed point in $(A_1^+)'$, and $x_g^+ \in A_1^+$. q.e.d.

In particular,

$$(1) \quad g_1^{-j_1}(g(A_1^+)) \subset A_{i_2}^{\text{sign}(j_2)} \quad \text{and} \quad g_2^{-j_2}g_1^{-j_1}(g(A_1^+)) \subset A_{i_3}^{\text{sign}(j_3)}.$$

3. Separating wedges

For a conical neighborhood $A \subset \mathbf{C}$ with bounding vector pair (u, v) , let

$$\Theta(A) = 2 \arcsin(\rho(u, v)/2),$$

and $t(A) = (u - v)$.

The horizontal plane through x is defined to be $H_x = \{y | y_3 = x_3\}$. Denoting the origin by o , we say that y is a horizontal vector if $y \in H_o$, that is $y_3 = 0$. $y, w \in H_x$ for some x if and only if $y - w$ is a horizontal vector.

Note that $t(A)$ is a horizontal vector. $\Theta(A)$ can be interpreted as the angle between the projection of its bounding vectors u and v onto H_o .

Define the wedge

$$W(A) = \left\{ w \in E \left| \begin{array}{l} B(u \boxtimes v, \frac{1}{w_3}w) \geq 0 \quad \text{if } B(w, w) \leq 0, \text{ or} \\ B(u \boxtimes v, x_w^+) \geq 0 \quad \text{if } B(w, w) > 0, \end{array} \right. \right\}.$$

See Figure 1.

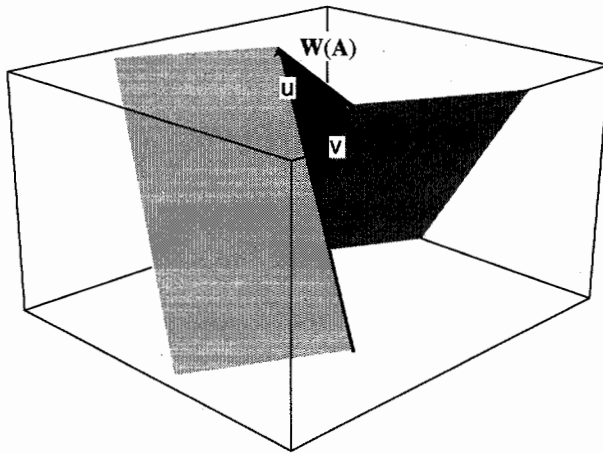


FIGURE 1. THE BOUNDARY OF A WEDGE.

The set of $y \in W(A)$ such that $B(y, y) \leq 0$ is a closed set bounded by A itself and (u, v) . For $y \in E$ such that $B(y, y) > 0$, denote the half-planes tangent to C and containing y as

$$P(y) = \{w \in E \mid B(w, w) > 0 \text{ and } x_w^+ = x_y^+\}.$$

$P(y) \subset W(A)$ if and only if $x_v^+ \in A$.

Define the *translated wedge*.

$$T(A) = [W(A) + t(A)].$$

(See [1] for more discussion.) By examining Figures 2–4 (see next page), we note that $T(A) \subset W(A)$. In fact, $t(A)$ is the unique horizontal vector v such that for all $k \geq 0$, $(W(A) + kv) \subset W(A)$.

Again citing Figures 2–4, we claim that if the interiors of conical neighborhoods A_1, A_2, \dots, A_n are mutually disjoint, then $T(A_i) \cap T(A_j) = \emptyset$, for $i \neq j$.

Lemma 2. *If G is a generalized Schottky group in G of rank n , then for all $1 \leq i \leq n$ there exist h_i and X_i such that X_i is a fundamental domain for the action of $\langle h_i \rangle$ on E , $L(h_i) = g_i$, $G = \langle g_1, g_2, \dots, g_n \rangle$, and $(E - X_i)$ is a submanifold of X_j for $i \neq j$.*

Proof. Because G is a generalized Schottky group, there are generators g_i and associated conical neighborhoods $A_i^\pm \subset C$ whose bounding vector pairs we denote (u_i^\pm, v_i^\pm) . A_i^\pm are chosen so that their interiors are disjoint and

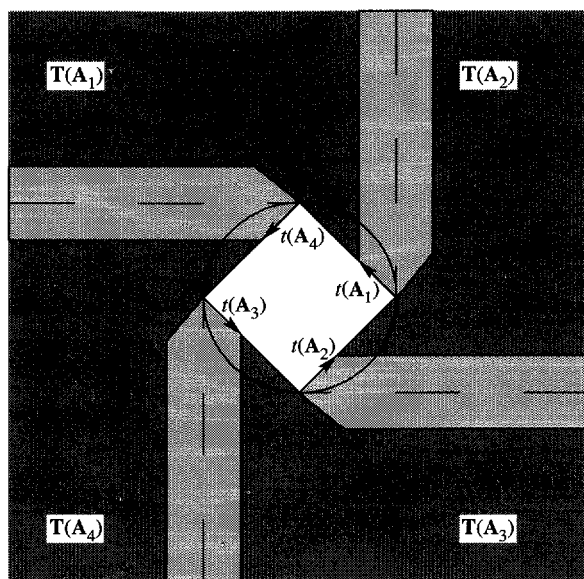


FIGURE 2. CROSS SECTION OF NONINTERSECTING TRANSLATED WEDGES ($x_3 = c > 0$).

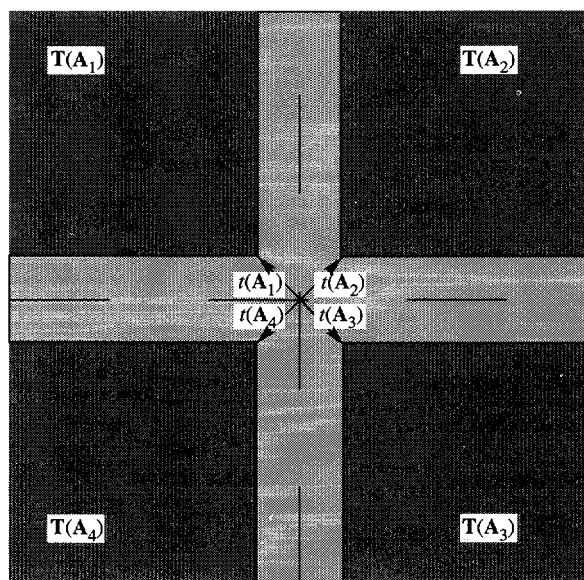


FIGURE 3. CROSS SECTION OF NONINTERSECTING TRANSLATED WEDGES ($x_3 = 0$).

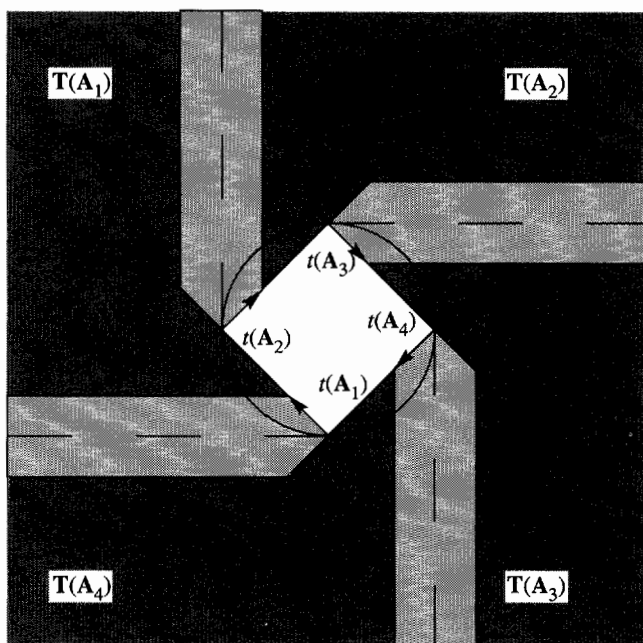


FIGURE 4. CROSS SECTION OF NONINTERSECTING TRANSLATED WEDGES ($x_3 = c < 0$).

$$g_i(A_i^-) = \text{cl}(C - A_i^+).$$

We claim that

$$g_i(W(A_i^-)) = \text{cl}(E - W(A_i^+)),$$

or more directly,

$$(2) \quad g_i(W(A_i^-)) = W(g_i(A_i^-)).$$

Remember that the set of $y \in W(A_i^-)$ such that $B(y, y) \leq 0$ is a closed set bounded by A_i^- itself and $\langle u_i^-, v_i^- \rangle$.

$$g_i(\langle u_i^-, v_i^- \rangle) = \langle g_i(u_i^-), g_i(v_i^-) \rangle = \langle u_i^+, v_i^+ \rangle,$$

since g_i takes elements of bounding vector pairs to scalar multiples of elements of bounding vector pairs. Certainly for y such that $B(y, y) \leq 0$, $y \in W(A_i^-)$ if and only if $g(y) \in W(g_i(A_i^-))$.

For $y \in W(A_i^-)$ such that $B(y, y) > 0$, it suffices to show that

$$(3) \quad g_i(P(y)) = P(g_i(y)).$$

In fact, it is enough to show that (3) is true for at least one y .

If g_i is hyperbolic, then $\mathbf{P}(x_{g_i}^o)$ can be written as

$$\{w \in \mathbf{E} | w = m(x_{g_i}^+) + n(x_{g_i}^o) \text{ for } m \in \mathfrak{R} \text{ and } n \in \mathfrak{R}^+\}.$$

The eigenvalue associated with $x_{g_i}^+$ may be negative, but the eigenvalue associated with $x_{g_i}^o$ is always 1, because $g \in \mathbf{G}$. $\mathbf{P}(x_{g_i}^o)$ is invariant under the action of g_i .

For parabolic g_i there exists a $j \in \mathbf{G}^o$ such that jg_i is hyperbolic. For y such that $B(y, y) > 0$ we know that

$$jg_i(\mathbf{P}(y)) = \mathbf{P}(jg_i(y)),$$

and

$$j^{-1}(jg_i(\mathbf{P}(y))) = \mathbf{P}(j^{-1}(jg_i(y))),$$

so that (3) is true for parabolic elements in \mathbf{G} .

Now choose

$$h_i(x) = g_i(x) + [-g_i(t(\mathbf{A}_i^-)) + t(\mathbf{A}_i^+)].$$

Then $h_i(t(\mathbf{A}_i^-)) = t(\mathbf{A}_i^+)$, and

$$\mathbf{X}_i = \text{cl}[\mathbf{E} - \mathbf{T}(\mathbf{A}_i^-) - \mathbf{T}_i(\mathbf{A}_i^+)]$$

is a fundamental domain for the action of $\langle h_i \rangle$ on \mathbf{E} since all of the translated wedges are distinct by the previous discussion. Further, $(\mathbf{E} - \mathbf{X}_i)$ is a 3-dimensional submanifold of \mathbf{X}_j . q.e.d.

Let $\mathbf{X} = (\bigcap_{i \in I} \mathbf{X}_i)$. Before showing that \mathbf{X} is the fundamental domain for the action of Γ on \mathbf{E} , we will prove the following technical lemma. For $g \in \mathbf{G}^o$ and $p \in \mathbf{E}$, We define $\mathbf{S}_g(p)$ to be the plane containing p and parallel to

$$\mathbf{S}_g = \langle x_g^o, x_g^+ \rangle.$$

Lemma 3. For ε -hyperbolic $g \in \mathbf{G}$,

$$(4) \quad \mathbf{B}\left(g(p), \frac{\varepsilon\delta}{2}\right) \cap \mathbf{S}_g(g(p)) \subset g(\mathbf{B}(p, \delta) \cap \mathbf{S}_g(p)).$$

Proof. It is sufficient to consider $p = 0$, and $\mathbf{S}_g(p) = \mathbf{S}_g$. Let \mathbf{Q} denote the rectangle in $\mathbf{B}(p, \delta) \cap \mathbf{S}_g$ whose four vertices are the four of $[\langle x_g^o \rangle \cup \langle x_g^+ \rangle] \cap \mathbf{B}(o, \delta)$. Note that

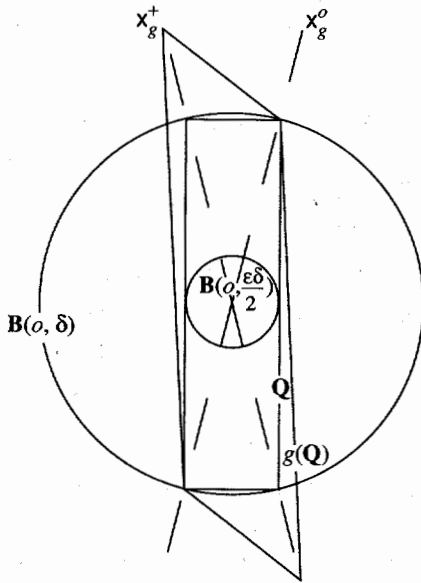


FIGURE 5. S_g .

$$B(o, \epsilon\delta/2) \cap S_g \subset Q \cap S_g.$$

See Figure 5.

g is a linear map which fixes the vertices of Q on $\langle x_g^o \rangle$ and sends the vertices of Q on $\langle x_g^+ \rangle$ to points on $\langle x_g^+ \rangle$ further from the origin. Thus, $Q \subset g(Q)$ and (4) follows. q.e.d.

The estimate in Lemma 3 is a lower bound of the *compression* by g parallel to S_g . Note that (4) is independent of λ_g .

Theorem 4. *If $G \subset \mathbf{G}$ is a free discrete group, then there exists $\Gamma \subset \mathbf{G} \times \mathbf{V}$ which acts properly discontinuously on \mathbf{E} such that $L(\Gamma) = G$.*

Proof. By Theorem 3, G is a generalized Schottky group.

We can choose g_i, h_i , and X_i as in Lemma 2. For $I = \{1, 2, \dots, n\}$, it suffices to show that for the 3-dimensional manifold $X = (\bigcap_{i \in I} X_i)$ with boundary is a fundamental domain for the action of Γ on \mathbf{E} .

From the construction of X it is apparent that no two distinct points in the interior of X are Γ -equivalent. It remains to show that every element of \mathbf{E} is Γ -equivalent to some point in X .

Assume that there is a p not Γ -equivalent to any $y \in X$. Certainly p is contained in one of the translated wedges $T(A_i^\pm)$ since their union is the complement of X . p is also Γ -equivalent to elements in all of the translated wedges $T(A_i^\pm)$. Thus, we may assume that $p \in T(A_1^+)$ and that

$\Theta(A_1^+) \leq \pi/2$, since the sum of the $\Theta(A_i^\pm)$'s is not more than 2π .

Let $X_0 = X$ and

$$X_{n+1} = \left[X_n \cup \left(\bigcup_{i=1}^n (h_i(X_n) \cup h_i^{-1}(X_n)) \right) \right].$$

This is a sequence of domains for which $\rho(p, X_{n+1}) \leq \rho(p, X_n)$. We can define $\gamma_n \in \Gamma$ such that $\gamma_n(X) \subset X_n$ and

$$\rho(p, X_n) = \rho(p, \gamma_n(X)).$$

γ_n has word length n as a reduced word in the free group Γ . For $n \geq 1$, $\gamma_n(X) \subset T(A_1^+)$ so the leading term of γ_n must be h_1 .

Let (u_i^+, v_i^+) be the bounding vector pair for A_i^+ , and let

$$w = u_1^+ - v_1^+.$$

If $v \in H_0$ is parallel to a ray lying in $T(A_1^+)$, then the angle between w and v is less than or equal to $\pi/4$.

Let $L_n \subset H_p$ be the line closest to p , which is Euclidean perpendicular to w and bounds a half-plane in H_p containing the component of the complement of $X_n \cap H_p$ which contains p . Note that

$$d_n = \rho(p, L_n) \geq \rho(p, X_n)$$

and $d_{n+1} \leq d_n$. To arrive at a contradiction it suffices to show that $(d_n - d_{n+1})$ is bounded away from 0. See Figure 6.

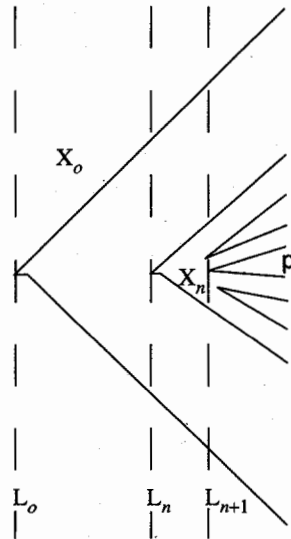


FIGURE 6. H_p .

There exists a $\delta > 0$ such that $\mathbf{B}(y, \delta) \subset \mathbf{X}_1$ for all $y \in \mathbf{X}$. Thus, $(d_1 - d_0) > \delta$. Choose

$$\varepsilon = \min\{\sqrt{2} \sin(\frac{1}{2}\Theta(\mathbf{A}_i^\pm))\}.$$

For $n \geq 1$, first suppose that γ_n is ε -hyperbolic. For every $x \in \mathbf{E}$, $S_{L(\gamma_n)}(\gamma_n(x)) \cap \mathbf{H}_p$ is parallel to a ray lying entirely within $\mathbf{T}(\mathbf{A}_1^+) \cap \mathbf{H}_p$ by Lemma 1. Every ball $\mathbf{B}(y, \delta)$ cannot be compressed by more than a factor of $\varepsilon/2$ in a direction parallel to $S_{L(\gamma_n)}$ by the action of γ_n . Since the angle between $S_{L(\gamma_n)}(\gamma_n(x)) \cap \mathbf{H}_p$ and the normal to L_n in \mathbf{H}_p is at most $\pi/4$,

$$d_{n+1} \leq (d_n - \varepsilon\delta/2\sqrt{2}).$$

Now suppose γ_n is not ε -hyperbolic. There exists an $f_n \in \Gamma$ with word length ≤ 2 such that $f_n\gamma_n$ is ε -hyperbolic and has word length $n + 1$ if f_n has word length 1 or $n + 2$ if f_n has word length 2. It is enough to consider f_n having length 2.

f_n can be written as $h_{a_n}h_{b_n}$, where h_{a_n} and h_{b_n} are generators of Γ or their inverses. $S_{L(f_n\gamma_n)}(f_n\gamma_n(x)) \cap \mathbf{H}_p$ is parallel to a ray lying entirely within $\mathbf{T}(\mathbf{A}_a^+) \cap \mathbf{H}_p$. δ -balls are not compressed by more than a factor of $\varepsilon/2$ in the direction parallel to $S_{L(f_n\gamma_n)}$ by the action of $f_n\gamma_n$.

We can define the *compression factor* for $g \in \mathbf{G}$,

$$C_g = \min_{v \in S^2} \{\|g(v)\|/\|v\|\},$$

which is positive for all $g \in \mathbf{G}$. Let C_Γ be the minimum of the compression factors of the g_i 's. Then $C_{f_n} \leq C_\Gamma^2$.

Thus, δ -balls are compressed by at most a factor of $C_\Gamma^2\varepsilon/2$ in the direction parallel to $[L(f_n^{-1}(S_{L(f_n\gamma_n)})]$ by the action of γ_n . From (1), $S_{L(f_n\gamma_n)}(\gamma_n(x)) \cap \mathbf{H}_p$ is parallel to a ray lying entirely within $\mathbf{G}(\mathbf{A}_1^+) \cap \mathbf{H}_p$. In this case,

$$d_{n+1} \leq (d_n - C_\Gamma^2\varepsilon\delta/2\sqrt{2}).$$

There must be an $m \leq 2\sqrt{2}d_0/(C_\Gamma^2\varepsilon\delta)$ such that $p \notin \mathbf{X}_n$ but $p \in \mathbf{X}_{n+1}$.

4. The end

Theorem 4 proves Theorem 2 is one direction, and in this section we will prove the other direction of Theorem 2.

If $G = L(\Gamma)$ for some free $\Gamma \subset \text{Aff}(\mathfrak{A}^3)$, then G is conjugate to a subgroup of $\mathbf{O}(2, 1)$ by [3]. Further, G is a discrete subgroup of $\mathbf{O}(2, 1)$ by [6].

Consider $G \subset \mathrm{PGL}_2(\mathfrak{R})$, and assume that $G \cap (\mathbf{O}(2, 1) - \mathbf{G}) \neq \emptyset$. There must be elements in $G \cap (\mathbf{O}(2, 1) - \mathbf{G})$ which have three distinct real eigenvalues and do not have 1 as an eigenvalue, but rather -1 is an eigenvalue. By an observation of Hirsch (see [3]), affine elements whose linear parts do not have 1 as an eigenvalue have fixed points. This contradicts the assumption that G acts properly on E . Thus, G must be conjugate to a finitely generated free discrete subgroup in \mathbf{G} .

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